Stochastic Loewner Equation and Critical Phenomena in 2D

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A first mystery: S, L, E ...

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Stochastic Evolution from the Loewner Equation

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Stochastic Evolution from the Loewner Equation derived by Schramm

Names and rewards ...

- Oded Schramm: the 2002 Clay Research Institute Award

- Stanislav Smirnov: the 2001 Clay Research Institute Award

... and some more

- Greg Lawler: the 2006 George Polya Award (with Schramm and Werner)

- Wendelin Werner: the 2006 Fields Medal

The (everyday) Loewner equation

- Karel Löwner, Karl Löwner, Charles Loewner: 1893 - 1968

- Loewner equation (1923) - used to prove the Bieberbach conjecture ($|a_n| \leq n$ for univalent functions) by Louis de Branges (1985)

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$$\boxed{\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z)-\xi_t}, \quad g_0(z) = z}$$

Loewner equation: evolution of conformal maps

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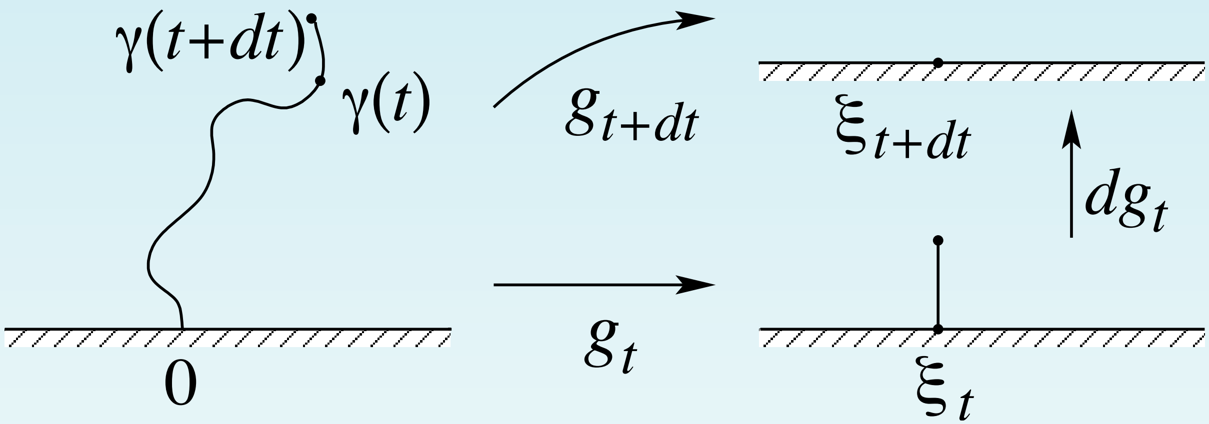
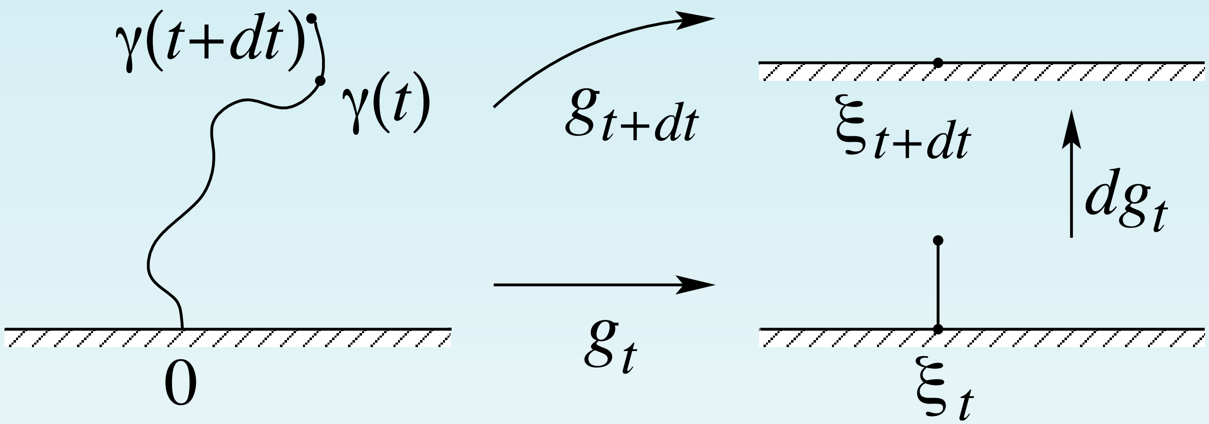


$$dg_t(w) = \xi_t + \sqrt{(w - \xi_t)^2 + 4dt},$$

Loewner equation: evolution of conformal maps



$$dg_t(w) = \xi_t + \sqrt{(w - \xi_t)^2 + 4dt},$$

$$g_{t+dt}(z) = \xi_t + \sqrt{(g_t(z) - \xi_t)^2 + 4dt} \approx g_t(z) + \frac{2dt}{g_t(z) - \xi_t}.$$

Standard maps

- Upper half-plane: chordal case

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- Stripe $\{z \in \mathbb{C} : |\Im z| \leq \pi\Delta\}$: dipolar case

$$\dot{g}_t(z) = \frac{\Delta^{-1}}{\tanh[(z - \xi_t)/\Delta]}$$

Schramm-Loewner evolution

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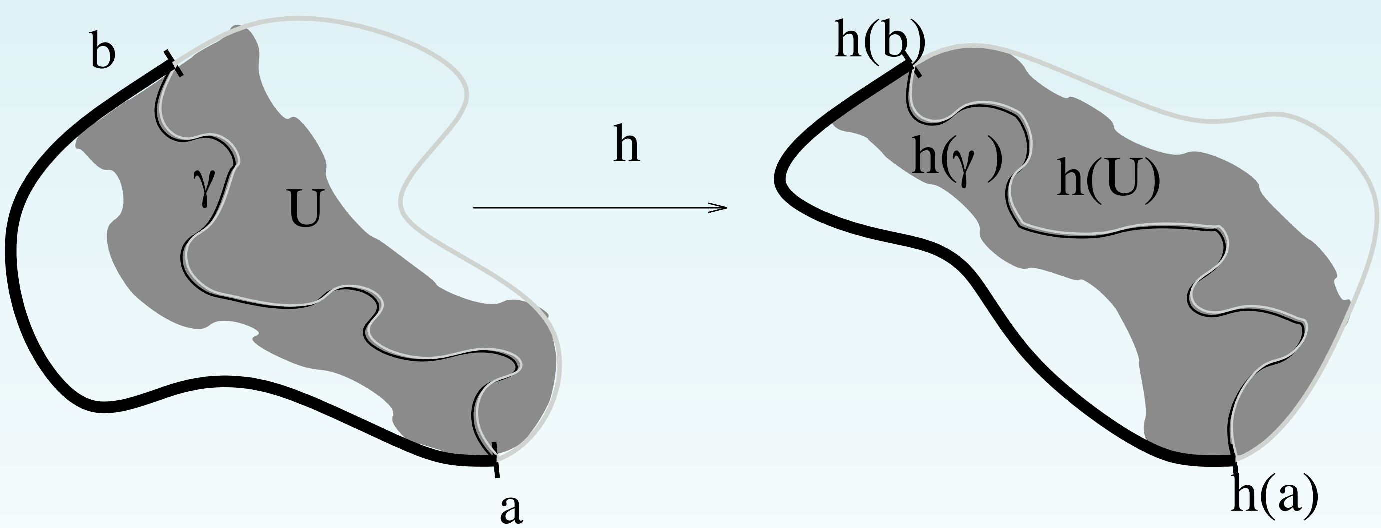
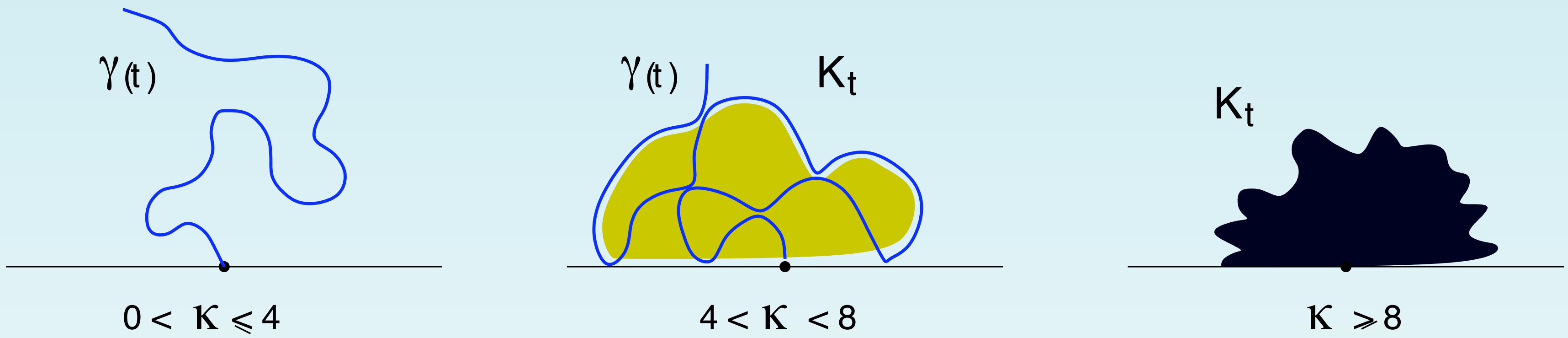
$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \; g_0(z) = z \; (\mathsf{SLE}_k).$$



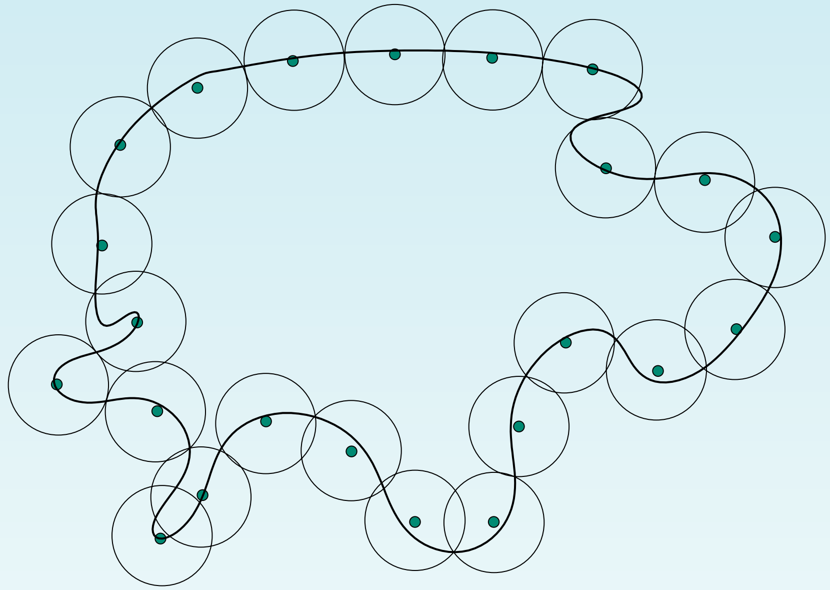
The phases of SLE



Fractal dimension of the trace d_f:

$$d_f(\kappa) = \begin{cases} 1 + \frac{\kappa}{8} & \text{for } \kappa \leq 8, \\ 2 & \text{for } \kappa \geq 8. \end{cases}$$

Scaling properties of SLE traces



$$N_\epsilon \sim \epsilon^{-d_f(\kappa)}, \quad \text{multifractal spectrum}$$

2D critical phenomena and CFT

- 2D Ising model :

$$Z[\beta, h] = \sum_{S_i=\pm 1} \exp\left[-\beta\left(\sum_{\langle i,j\rangle} S_i S_j + h\sum_i S_i\right)\right]$$

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- Onsager (1934) - Baxter (transfer matrix)
- McCoy and Wu (1972), Jimbo-Miwa-Sato-Ueno (1980) - Painlevé transcendents: conformal invariance as linearization

SLE and lattice statistical models

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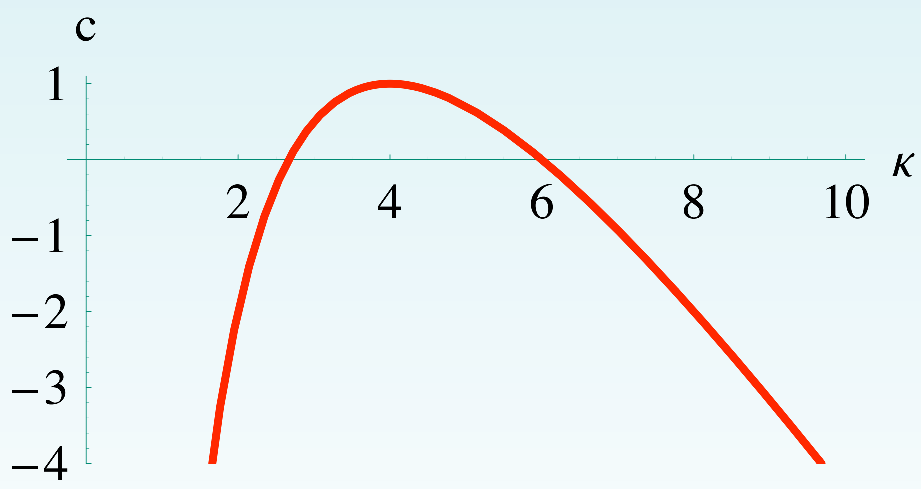
$$c_\kappa = \frac{(8-3\kappa)(\kappa-6)}{2\kappa} = 1 - 3\frac{(\kappa-4)^2}{2\kappa}, \;\; c_\kappa = c_{\kappa'}, \quad \kappa' = \frac{16}{\kappa}.$$



Known and conjectured correspondences

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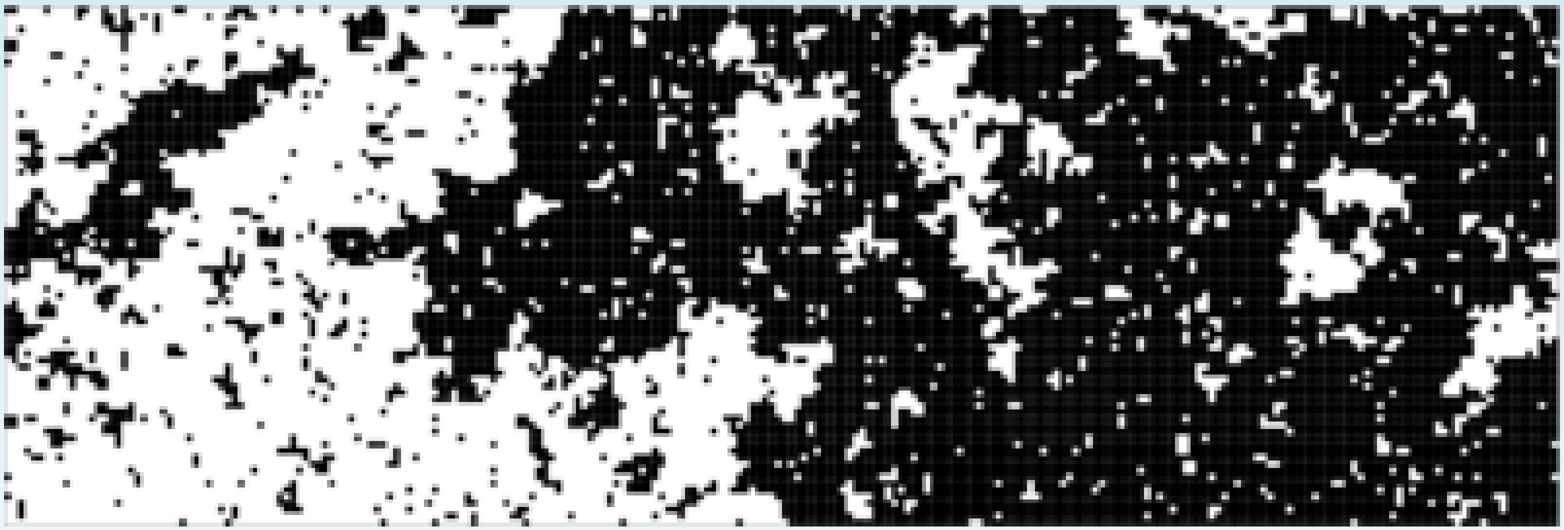
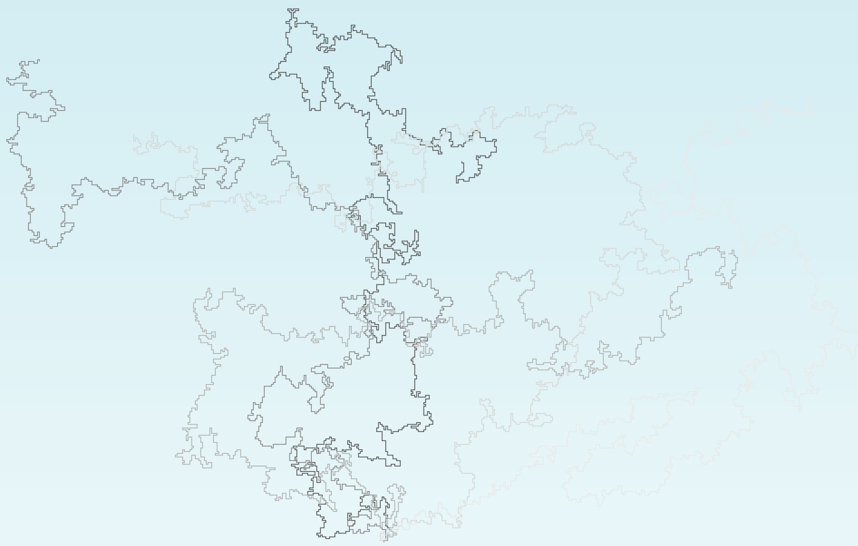
Lattice model	κ	c_κ
Loop-erased random walk	2	-2
Self-avoiding random walk	$8/3$	0
Ising model spin cluster boundaries	3	$1/2$
Dimer tilings	4	1
Harmonic explorer	4	1
Level lines of Gaussian field	4	1
Percolation cluster boundaries	6	0
Uniform spanning trees	8	-2

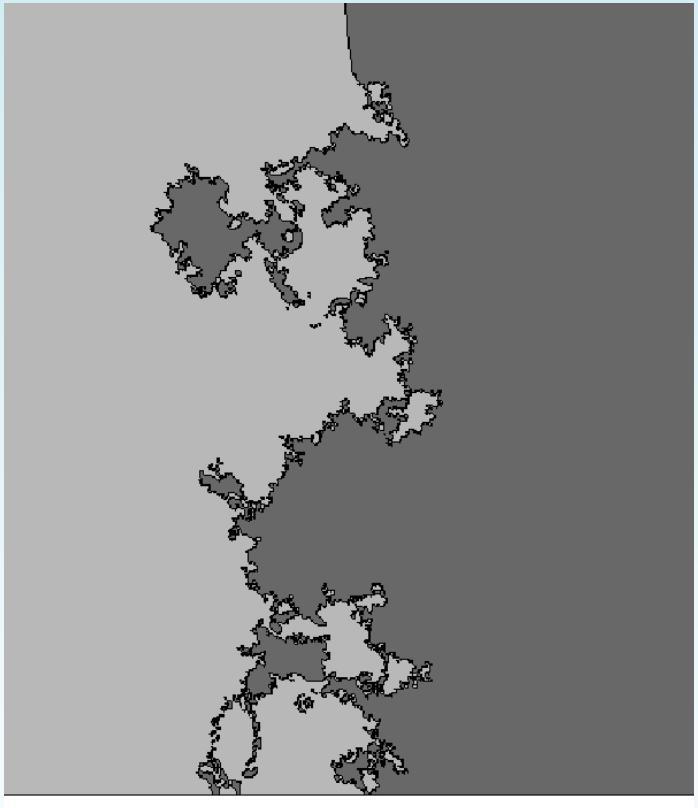
Ising clusters



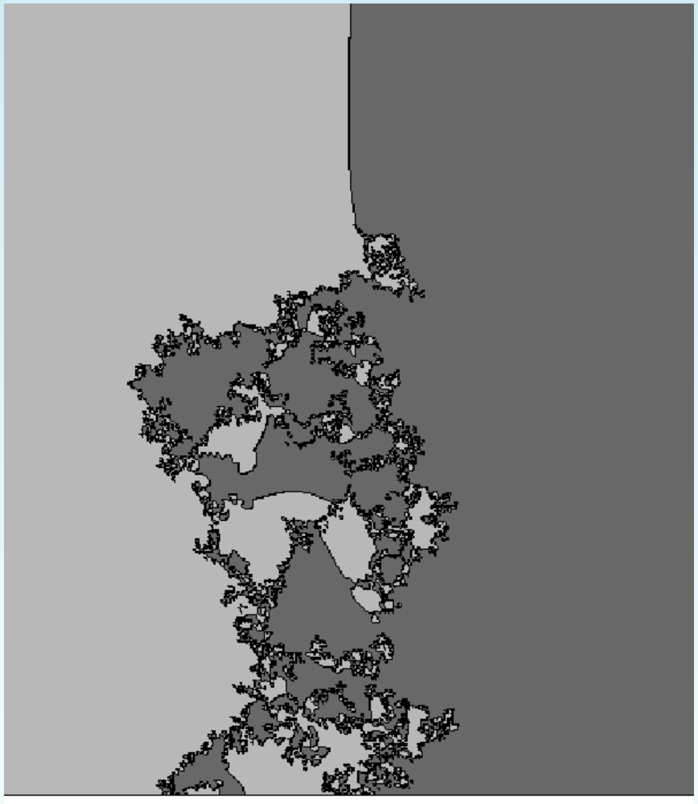
SAW random walk



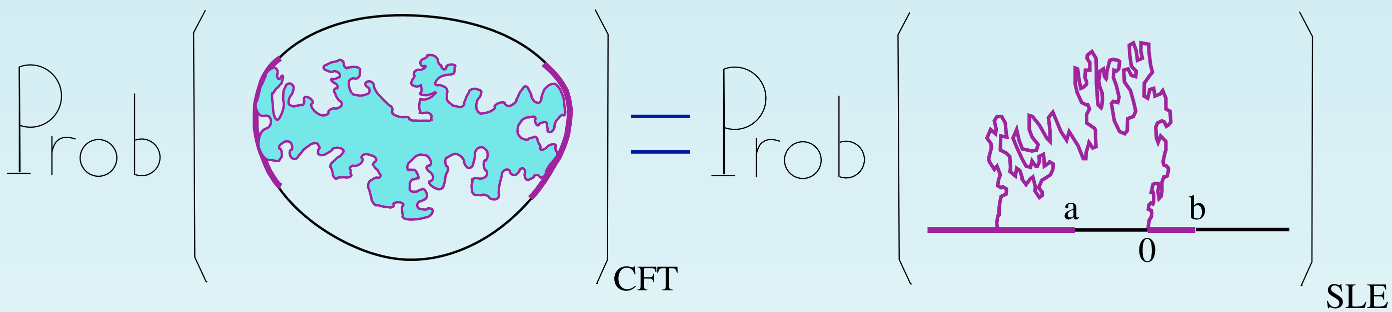
Gaussian field level lines



Percolation clusters



Cardy's formula for anisotropic percolation



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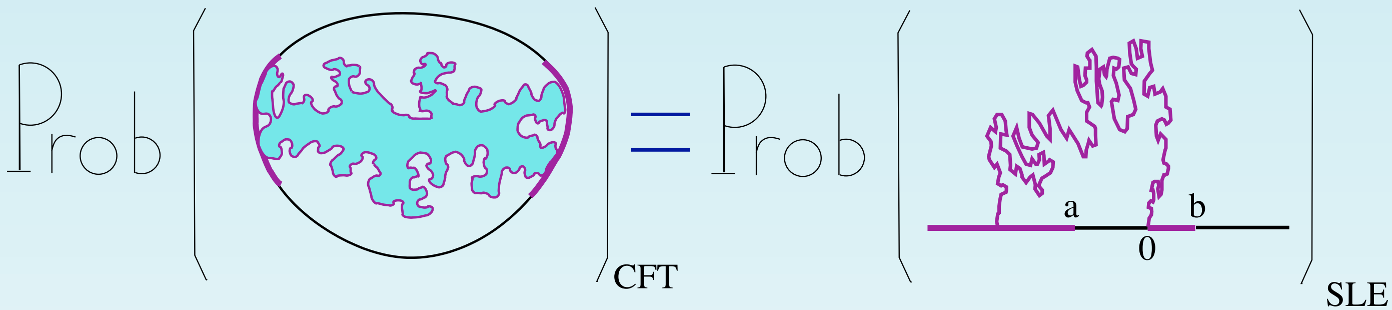
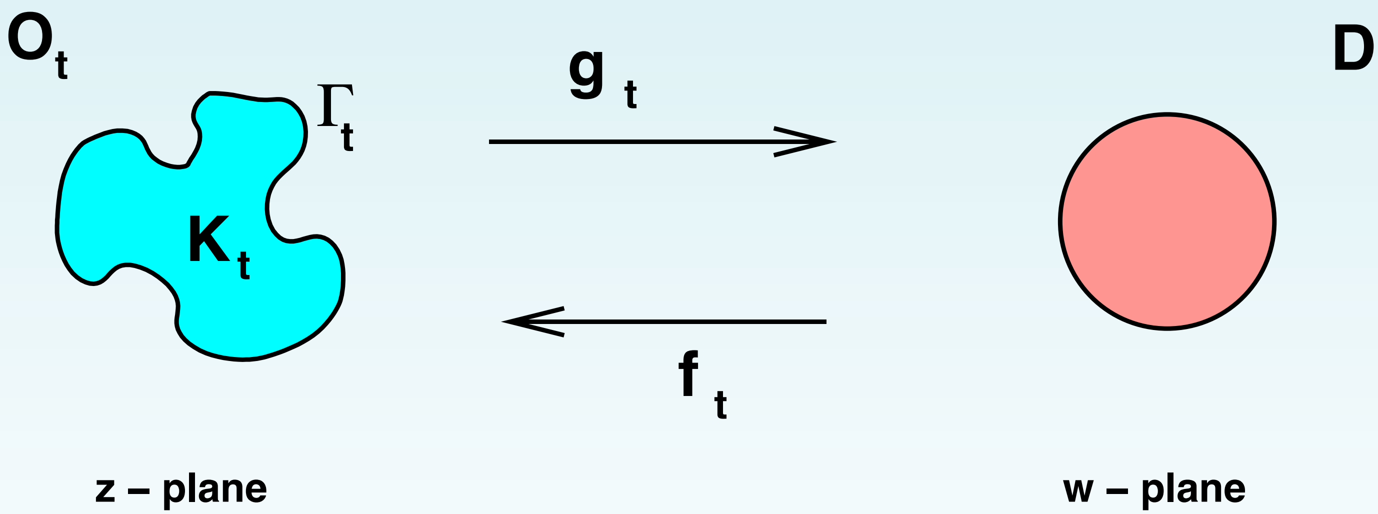


$$\mathbf{P}[\text{crossing}] = \frac{\Gamma\left(2-\frac{8}{\kappa}\right)}{\Gamma\left(2-\frac{4}{\kappa}\right)\Gamma\left(1-\frac{4}{\kappa}\right)} r^{1-4/\kappa} {}_2F_1\left(\frac{4}{\kappa}, 1-\frac{4}{\kappa}; 2-\frac{4}{\kappa}; r\right).$$

Radial SLE

$$\dot{g}_t(z) = -g_t(z)\,\frac{g_t(z) + e^{i\sqrt{\kappa}B(t)}}{g_t(z) - e^{i\sqrt{\kappa}B(t)}}$$



Multiple SLE

N driving forces ξ_i:

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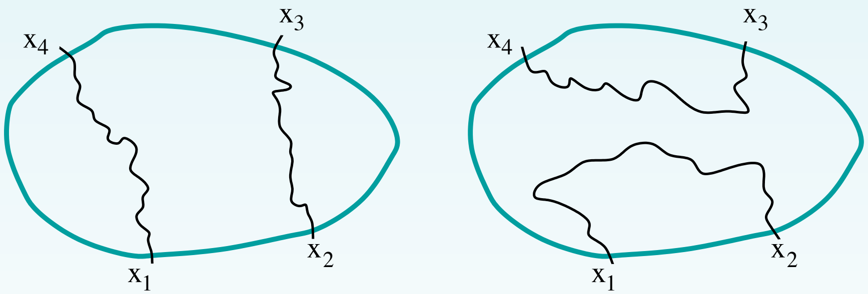
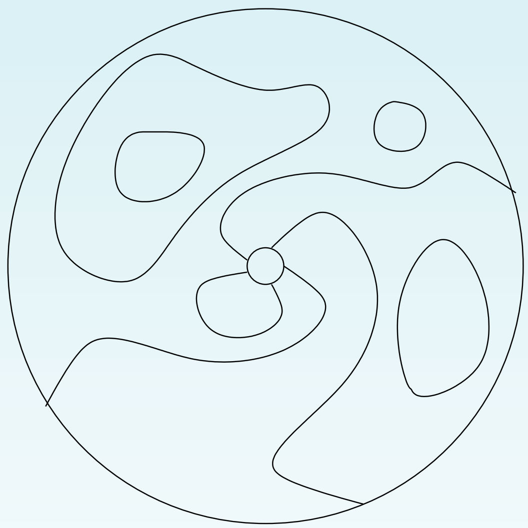
N-point correlation functions



Multiple radial SLE's

J. Cardy, *Stochastic Loewner evolution and Dyson's circular ensembles*, J. Phys. A: Math. Gen. **36**, L379 (2003); arXiv: math-ph/0301039.



Radial SLE and Calogero-Sutherland model

Simple-pole solution of Kadomtsev-Petviashvilii integrable hierarchy

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$$P_{\rm eq}(\{\theta_j\}) \propto \prod_{1\le j<k\le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{\beta}, \quad \beta = 4/\kappa \quad \text{(Dyson)}.$$

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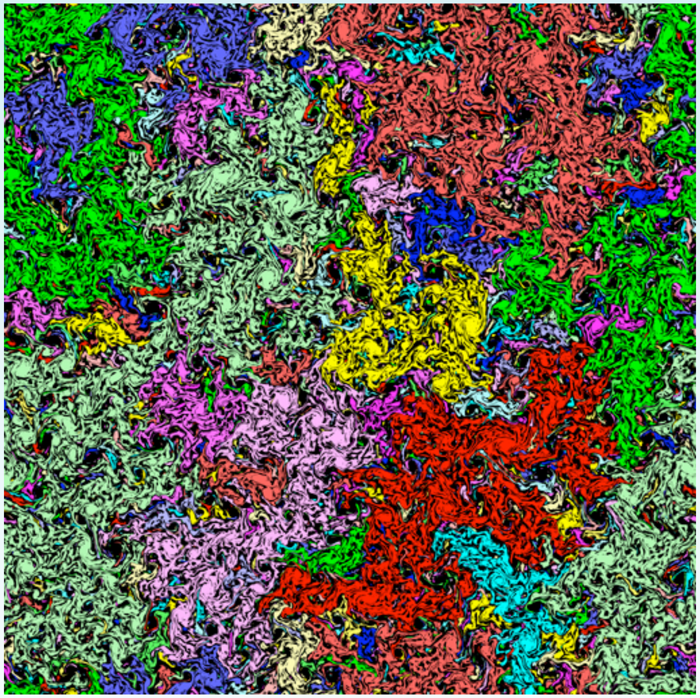
Correlations functions of N radial SLE solve the Calogero-Sutherland model:

$$\mathcal{H} = -\frac{\kappa}{2} \sum_j \frac{\partial^2}{\partial \theta_j^2} + \frac{2-\kappa}{2\kappa} \sum_{j<k} \frac{1}{\sin^2(\theta_j - \theta_k)/2} - \frac{N(N-1)}{2\kappa}$$

Zero vorticity lines in 2D turbulence and SLE$_6$

D. Bernard, G. Boffetta, A. Celani, G. Falkovich, *Conformal invariance in two-dimensional turbulence*, Nature Physics **2**, 124 (2006).



Domain walls in Ising spin glasses

C. Amoruso, A. K. Hartmann, M. B. Hastings, and M. A. Moore, *Conformal invariance and SLE in two-dimensional Ising spin glasses*, arXiv: cond-mat/0601711.

Conformal invariance of domain walls:

$$d_f = \frac{5}{4}, \quad \kappa = 2, \quad \text{Loop-erased RW}$$

SLE chains and 2D growth processes

$N \to \infty$ generalization of N radial SLE:

$$\frac{\partial}{\partial t} g_t(z) = -g_t(z) \oint \frac{\rho_t(u) du}{2i\pi u} \Big(\frac{g_t(z) + u}{g_t(z) - u}\Big)$$

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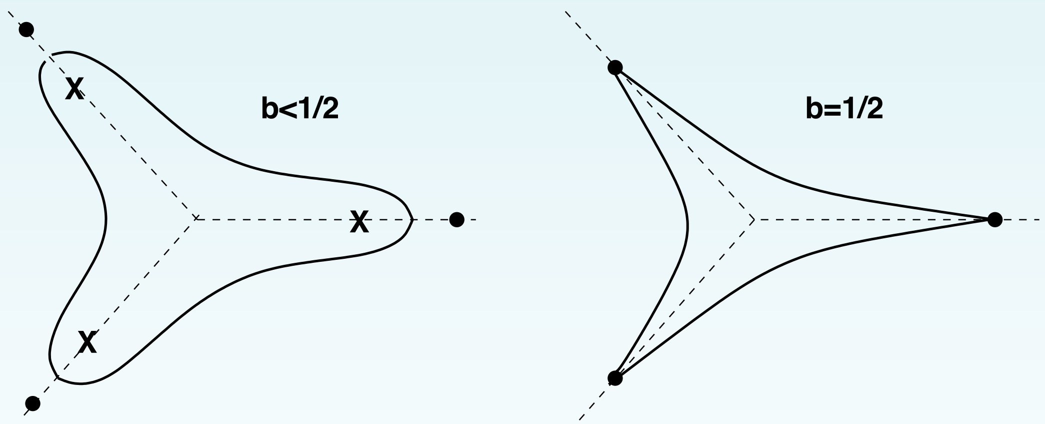

2D growth under harmonic forces: DLA and LG

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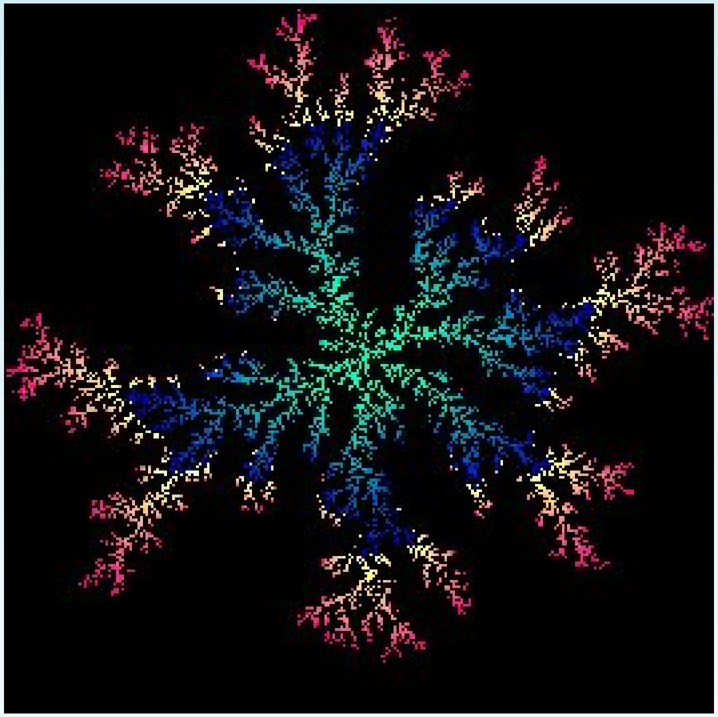
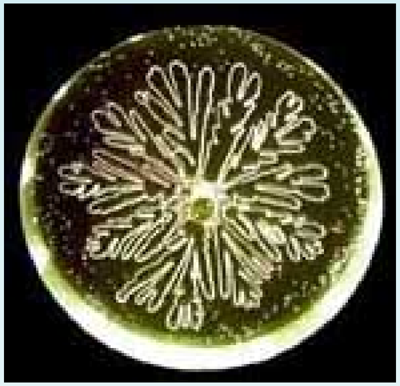
- Averaged process: Laplacian Growth

Radial diffusion limited aggregation



Radial laplacian growth (idealized Hele-Shaw flows)



$$V_n = -\vec{\nabla}_n p$$
$$\Delta p = 0 \quad \text{outside}$$

Resolving finite-time singularities of Hele-Shaw flows (Saffman,Taylor, Sakai, Kadanoff, Bensimon, Howison, King, Tanveer, Crowdy, ...)

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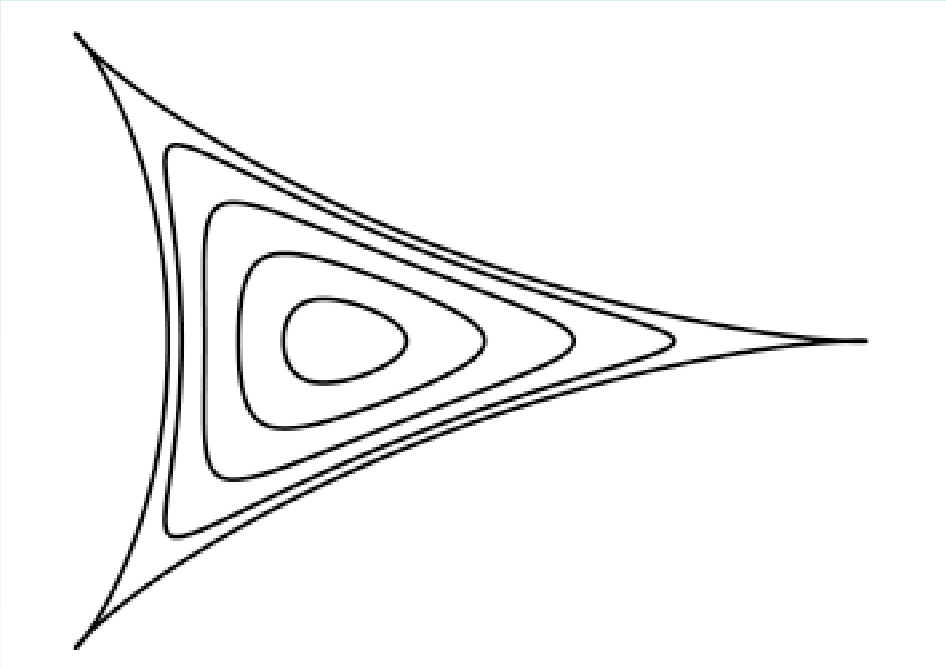

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